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Computational construction of irreducible W-graphs for types E_6 and E_7

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ABSTRACT

The concept of W-graph was introduced in the influential paper [David Kazhdan, George Lusztig, Representations of Coxeter groups and Hecke algebras, Invent. Math. 53 (1979) 165–184] of Kazhdan and Lusztig. If the underlying Coxeter group is the symmetric group, then Kazhdan and Lusztig showed that every irreducible representation arises from a left cell and, hence, is given by a W-graph. This is the optimal picture that can hope for. For other types of Coxeter groups, the representations arising from left cells are no longer irreducible. However, Gyoja [A. Gyoja, On the existence of a W-graph for an irreducible representation of a Coxeter group, J. Algebra 86 (1984) 422–438. [3]] proved, by a general argument, that every irreducible representation of a Hecke algebra associated with a finite Coxeter group is given by a W-graph, but this is a pure existence result, and the question remained open of how to construct such W-graphs explicitly. In [R.B. Howlett, Yunchuan Yin, Inducing W-graphs, Math. Z. 244 (2003) 415–431] we provided a general method for producing W-graphs, by induction of W-graphs from parabolic subgroups, and then [Yunchuan Yin, Irreducible W-graphs for type D_4 and D_5, Comm. Algebra 34 (2006) 547–565. [7]] we constructed all the irreducible W-graphs for type D_4 and D_5 by hand calculation. In this paper we will show that the algorithm introduced in [R.B. Howlett, Yunchuan Yin, Inducing W-graphs, Math. Z. 244 (2003) 415–431] is sufficiently powerful to construct explicit W-graphs for all irreducible representations of Hecke algebras of type E_6 and E_7. The computer algebra system “Magma” was used for the calculations.

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1. Preliminaries

Let W be a Coxeter group with S the set of simple reflections, and let $\mathscr{H}$ be the corresponding Hecke algebra. We use a variation of the definition given in [5], taking $\mathscr{H}$ to be an algebra over $\mathcal{A}=\mathbb{Z}[q^{-1},q]$, the ring of Laurent polynomials with integer coefficients in the indeterminate q, having an $\mathcal{A}$-basis $\{T_w \mid w \in W\}$ satisfying

$$T_sT_w = \begin{cases} T_{sw} & \text{if } \ell(sw) > \ell(w), \\ T_{sw} + (q-q^{-1})T_w & \text{if } \ell(sw) < \ell(w), \end{cases}$$

for all $w \in W$ and $s \in S$. We also define $\mathcal{A}^+ = \mathbb{Z}[q]$, the ring of polynomials in q with integer coefficients, and let $a \mapsto \bar{a}$ be the involutory automorphism of $\mathcal{A}$ such that $\bar{q} = q^{-1}$. This involution on $\mathcal{A}$ extends to an involution on $\mathscr{H}$ satisfying $\overline{T_s} = T_s^{-1} = T_s + (q^{-1}-q)$ for all $s \in S$. This gives $\overline{T_w} = T_{w^{-1}}^{-1}$ for all $w \in W$.

For each $J \subseteq S$ define $W_J = \langle J \rangle$, the corresponding parabolic subgroup of W, and let $D_J = \{w \in W \mid \ell(ws) > \ell(w) \text{ for all } s \in J\}$, the set of minimal coset representatives of W/W_J. Let $\mathscr{H}_J$ be the Hecke algebra associated with W_J. As is well known, $\mathscr{H}_J$ can be identified with a subalgebra of $\mathscr{H}$.

1.1. W-graphs

Modifying the definitions in [5] to suit our definition of the Hecke algebra, a W-graph is a set Γ (the vertices of the graph) with a set Θ of two-element subsets of Γ (the edges) together with the following additional data: for each vertex γ we are given a subset I_γ of S, and for each ordered pair of vertices δ, γ we are given an integer $\mu(\delta,\gamma)$ which is nonzero if and only if $\{\delta,\gamma\} \in \Theta$. These data are subject to the requirement that $\mathcal{A}\Gamma$, the free $\mathcal{A}$-module on Γ, has an $\mathscr{H}$-module structure satisfying

$$T_s\gamma = \begin{cases} -q^{-1}\gamma & \text{if } s \in I_\gamma, \\ q\gamma + \sum_{\{\delta\in\Gamma \mid s\in I_\delta\}} \mu(\delta,\gamma)\delta & \text{if } s \notin I_\gamma, \end{cases} \tag{1}$$

for all $s \in S$ and $\gamma \in \Gamma$. If τ_s is the $\mathcal{A}$-endomorphism of $\mathcal{A}\Gamma$ such that $\tau_s(\gamma)$ is the right-hand side of Eq. (1) then this requirement is equivalent to the condition that for all $s,t \in S$ such that st has finite order,

$$\underbrace{\tau_s\tau_t\tau_s\ldots}_{m \text{ factors}} = \underbrace{\tau_t\tau_s\tau_t\ldots}_{m \text{ factors}},$$

where m is the order of st.

To avoid over-proliferation of symbols, we shall use the name of the vertex set of a W-graph to also refer to the W-graph itself. We call I_γ the *descent set* of the vertex $\gamma \in \Gamma$, and we call $\mu(\delta,\gamma)$ and $\mu(\gamma,\delta)$ the *edge weights* associated with the edge $\{\delta,\gamma\}$.

Given a W-graph Γ we define

$$\Gamma_s^- = \{\gamma \in \Gamma \mid s \in I_\gamma\},$$
$$\Gamma_s^+ = \{\gamma \in \Gamma \mid s \notin I_\gamma\}.$$

The following lemma is well known.

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Lemma 1.1. *(See Deodhar [1, Lemma 3.2].) Let $J \subseteq S$ and $s \in S$, and define*

$$D^-_{J,s} = \{d \in D_J \mid \ell(sd) < \ell(d)\},$$
$$D^+_{J,s} = \{d \in D_J \mid \ell(sd) > \ell(d) \text{ and } sd \in D_J\},$$
$$D^0_{J,s} = \{d \in D_J \mid \ell(sd) > \ell(d) \text{ and } sd \notin D_J\},$$

so that D_J is the disjoint union $D^-_{J,s} \cup D^+_{J,s} \cup D^0_{J,s}$. Then $sD^+_{J,s} = D^-_{J,s}$, and if $d \in D^0_{J,s}$ then $sd = dt$ for some $t \in J$.

1.2. Construction of induced W-graphs

Following the notation and terminology of [4], we assume that Γ is a W_J-graph and M the corresponding induced $\mathscr{H}$-module.

Theorem 1.2. *(See [4, Theorem 5.1].) The module M has a unique basis*

$$\{C_{w,\gamma} \mid w \in D_J,\ \gamma \in \Gamma\}$$

such that $\overline{C_{w,\gamma}} = C_{w,\gamma}$ for all $w \in D_J$ and $\gamma \in \Gamma$, and

$$C_{w,\gamma} = \sum_{y \in D_J,\, \delta \in \Gamma} P_{y,\delta,w,\gamma} T_y \delta$$

for some elements $P_{y,\delta,w,\gamma} \in \mathcal{A}^+$ with the following properties:

(i) $P_{y,\delta,w,\gamma} = 0$ *if* $y \nleqslant w$;

(ii) $P_{w,\delta,w,\gamma} = \begin{cases} 1 & \text{if } \delta = \gamma, \\ 0 & \text{if } \delta \neq \gamma; \end{cases}$

(iii) $P_{y,\delta,w,\gamma}$ *has zero constant term if* $(y,\delta) \neq (w,\gamma)$.

The following recursive formula for the polynomials $P_{y,\delta,w,\gamma}$ is proved in [4]: $P_{y,\delta,w,\gamma} = P'_{y,\delta,w,\gamma} - P''_{y,\delta,w,\gamma}$, where

$$P'_{y,\delta,w,\gamma} = \begin{cases} P_{sy,\delta,v,\gamma} - qP_{y,\delta,v,\gamma} & \text{if } y \in D^+_{J,s}, \\ P_{sy,\delta,v,\gamma} - q^{-1}P_{y,\delta,v,\gamma} & \text{if } y \in D^-_{J,s}, \\ (-q-q^{-1})P_{y,\delta,v,\gamma} + \sum_{\theta \in \Gamma^+_t} \mu(\delta,\theta) P_{y,\theta,v,\gamma} & \text{if } y \in D^0_{J,s} \text{ and } \delta \in \Gamma^-_t, \\ 0 & \text{if } y \in D^0_{J,s} \text{ and } \delta \in \Gamma^+_t; \end{cases} \tag{2}$$

$$P''_{y,\delta,w,\gamma} = \sum_{\substack{(z,\theta) \prec (v,\gamma) \\ (z,\theta) \in \Lambda^-_s}} \mu(z,\theta,v,\gamma) P_{y,\delta,z,\theta}. \tag{3}$$

Given $y, w \in D_J$ and $\delta, \gamma \in \Gamma$ with $(y,\delta) \neq (w,\gamma)$, we define an integer $\mu(y,\delta,w,\gamma)$ as follows. If $y < w$ then $\mu(y,\delta,w,\gamma)$ is the coefficient of q in $-P_{y,\delta,w,\gamma}$, and if $w < y$ then it is the coefficient of q in $-P_{w,\gamma,y,\delta}$. If neither $y < w$ nor $w < y$ then

$$\mu(y,\delta,w,\gamma)=\begin{cases}\mu(\delta,\gamma) & \text{if } y=w,\\ 0 & \text{if } y\neq w.\end{cases}$$

We write $(y,\delta)\prec(w,\gamma)$ if $y<w$ and $\mu(y,\delta,w,\gamma)\neq 0$.

It is shown in Theorem 5.3 of [4] that the basis elements $C_{w,\gamma}$ can be identified with the vertices of a W-graph for the module M; we shall denote this W-graph by Λ. The descent set of the vertex $C_{w,\gamma}$ of Λ is

$$I(w,\gamma)=\{s\in S \mid \ell(sw)<\ell(w) \text{ or } sw=wt \text{ for some } t\in I_\gamma\}$$

and the edge weight for $((y,\delta),(w,\gamma))$ is $\mu(y,\delta,w,\gamma)$ (as defined above). Thus $\{C_{y,\delta},C_{w,\gamma}\}$ is an edge of Λ if and only if $\mu(y,\delta,w,\gamma)\neq 0$, and this occurs if and only if either $(y,\delta)\prec(w,\gamma)$ or $(w,\gamma)\prec(y,\delta)$, or $y=w$ and $\{\delta,\gamma\}$ is an edge of Γ.

We define

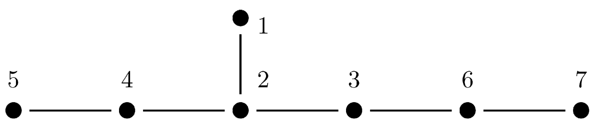
$$\Lambda_s^-=\{(w,\gamma)\in D_J\times\Gamma \mid s\in I(w,\gamma)\}=\{(w,\gamma) \mid w\in D_{J,s}^- \text{ or } w\in D_{J,s}^0 \text{ with } t\in I_\gamma\}.$$

Theorem 1.3. *(See [4, Theorem 5.2].) Let $w\in D_J$ and $\gamma\in\Gamma$. Then for all $s\in S$ such that $\ell(sw)>\ell(w)$ and $sw\in D_J$ we have*

$$T_sC_{w,\gamma}=qC_{w,\gamma}+C_{sw,\gamma}+\sum\mu(y,\delta,w,\gamma)C_{y,\delta}, \tag{4}$$

where the sum is over all $(y,\delta)\in\Lambda_s^-$ such that $(y,\delta)\prec(w,\gamma)$.

It is convenient to distinguish three kinds of edges of the W-graph Λ. Firstly, there is an edge from the vertex $C_{w,\gamma}$ to the vertex $C_{w,\delta}$ whenever there is an edge from γ to δ in Γ. We call these *horizontal* edges. Next, if $s\in S$ and w is in either $D_{J,s}^+$ or $D_{J,s}^-$ then there is an edge joining $C_{w,\gamma}$ and $C_{sw,\gamma}$. We call these *vertical* edges. All other edges are called *transverse*. For the purposes of machine calculation it was convenient to number the simple reflections of E_7 in such a way that numbers 1 to 5 generate D_5 and numbers 1 to 6 generate E_6. The numbering that was chosen is shown in the diagram below.



2. Using Magma [6] to induce W-graphs

In the terminology of Magma, each W-graph was represented as a record with four fields: basering, J, I and edges. Here basering is the ring in which the edge weights must lie, and can always be the ring of integers. The field J is the set of integers from 1 to n, where n is the rank of the Coxeter group in question: that is, n is 5, 6 or 7 for D_5, E_6 or E_7 (respectively). The field I is a sequence of subsets of J, giving the descent sets of the vertices of the graph. Thus, the number of terms of the sequence is the number of vertices, the ith term of the sequence gives the descent set of vertex number i. The field edges is a sequence of sets of pairs: the ith term of this sequence consists of all pairs $[j,m]$ such that $\{i,j\}$ is an edge of the W-graph, and the edge weight $\mu(i,j)$ is m.

For example, the W-graph of the exterior square of the reflection representation of E_6 is stored as a file with the following lines.

```
J:={1,2,3,4,5,6};
graphinfo:=recformat<J,I,edges,basering>;
gammaedges:=[{[2,1],[3,1]},
{[1,1],[5,1],[6,1]},
{[1,1],[4,1],[7,1]},
{[3,1],[8,1]},
{[9,1],[2,1]},
{[2,1],[10,-1],[9,1]},
{[3,1],[8,1],[10,1]},
{[4,1],[7,1],[11,1],[13,1]},
{[6,1],[5,1],[12,1],[14,1]},
{[7,1],[6,-1],[11,1],[14,-1]},
{[8,1],[10,1],[15,-1]},
{[9,1]},
{[8,1]},
{[9,1],[10,-1],[15,1]},
{[14,1],[11,-1]}];
IGamma:=
[
    { 1, 2 },
    { 1, 3 },
    { 1, 4 },
    { 1, 5 },
    { 1, 6 },
    { 2, 3 },
    { 2, 4 },
    { 2, 5 },
    { 2, 6 },
    { 3, 4 },
    { 3, 5 },
    { 3, 6 },
    { 4, 5 },
    { 4, 6 },
    { 5, 6 }].
```

Given a W-graph record wg for D_5, the following command will induce it to E_6:

```
ind:= induceWGraph(wg,action,[1,2,3,4,5]);
```

where action is a sequence of sequences, one for each simple reflection. The sequence corresponding to the simple reflection s is $[i_1, i_2, \ldots, i_N]$, where N is the number of positive roots, and applying s to the jth positive root produces the i_jth positive root, except when the jth positive root is the simple root corresponding to s. In this case i_j is set equal to $-j$. Thus the output of action is essentially just the permutation action of the simple reflections on the root system. In the example of E_6 above, the variable action takes the following value:

```
[-1,7,3,4,5,6,2,12,13,10,11,8,9,17,18,19,14,15,16,
 23,24,22,20,21,28,26,27,25,29,30,31,32,33,34,36,35],
[7,-2,8,9,5,6,1,3,4,16,15,12,13,14,11,10,22,18,19,
 20,21,17,26,27,25,23,24,30,29,28,31,32,33,35,34,36],
[1,8,-3,4,5,11,12,2,14,10,6,7,17,9,15,20,13,18,23,
 16,21,22,19,24,25,26,31,28,29,33,27,34,30,32,35,36],
[1,9,3,-4,10,6,13,14,2,5,11,17,7,8,21,16,12,24,19,
 20,15,22,23,18,25,29,27,28,26,32,31,30,34,33,35,36],
```

```
[1,2,3,10,-5,6,7,8,16,4,11,12,19,20,15,9,23,18,13,
 14,25,26,17,28,21,22,30,24,29,27,33,32,31,34,35,36],
[1,2,11,4,5,-6,7,15,9,10,3,18,13,21,8,16,24,12,19,
 25,14,27,28,17,20,30,22,23,32,26,31,29,33,34,35,36].
```

The function induceWGraph requires three arguments. The second of these is a sequence of sequences like action above, describing the action of the simple reflections of W on the root system. The first argument is a record describing a W_J-graph, where J is some subset of the set of simple reflections. The third argument is a sequence of length equal to the rank of W_J describing the embedding of J in S. Thus if this argument is $[1,2,3,4,5]$ (as above), the first simple reflection of W_J is identified with the first simple reflection of W, the second with the second, and so on. Given our conventions for numbering the simple roots of D_5 and E_6, we could use $[3,2,1,4,5]$, $[1,2,4,3,6]$ or $[4,2,1,3,6]$ instead of $[1,2,3,4,5]$ (but there is no point in doing so). For inducing from E_6 to E_7 the third argument of induceWGraph should be $[1,2,3,4,5,6]$ or $[1,2,4,3,6,5]$.

The output of induceWGraph is a record describing the W-graph $\mathrm{Ind}_J^S(\Gamma)$, where Γ is the W_J-graph corresponding to the first argument of induceWGraph.

3. Decomposing the induced graphs

Having computed an induced W-graph, as described in the previous section, it is a trivial task to break it into cells, since the Magma function getCells implements a suitable algorithm. Continuing the example from Section 2, after the command

```
ind:= induceWGraph(wg,action,[1,2,3,4,5]);
```

the command

```
cells:=getCells(ind);
```

returns a sequence of subsets of $\{i \mid 1 \leqslant i \leqslant d\}$, where d is the number of vertices of the induced W-graph $\mathrm{Ind}_J^S(\Gamma)$ corresponding to the record ind. Integers i and j lie in the same term of cells if and only if the ith and jth vertices of $\mathrm{Ind}_J^S(\Gamma)$ lie in the same cell. In this example it turns out that there are four cells, consisting of 15, 64, 81 and 110 vertices.

Next, the commands

```
cg1:=cellGraph(ind, cells[1]);
cg2:=cellGraph(ind, cells[2]);
cg3:=cellGraph(ind, cells[3]);
cg4:=cellGraph(ind, cells[4]);
```

produce W-graph records corresponding to the cells. This is simply an exercise in discarding the vertices that are not in the relevant cell and renumbering the ones that are left.

For all but a few of the irreducible characters of $W(E_6)$ and $W(E_7)$ there exist W-graphs occurring as cells in induced W-graphs. We now describe a more general procedure for extracting submodules from W-graph modules; this procedure enabled us to find W-graphs in all the remaining cases.

Given a W-graph datum (Γ, I, μ), let M be the free $\mathbb{Z}$-module with basis Γ, and for each $J \subseteq S$ let M_J be the free $\mathbb{Z}$-module with basis $\{\gamma \in \Gamma \mid I_\gamma = J\}$. Define $\phi_{JK} : M_K \to M_J$ by

$$\phi_{JK}\gamma = \sum_{\delta \in \Gamma_J} \mu(\delta, \gamma)\delta$$

for each $\gamma \in \Gamma_K$. Observe that if $v \in M_K$ and $s \notin K$ then

$$(T_s - q)v = \sum_{\{J \mid s \in J\}} \phi_{JK} v,$$

and it follows that if $(N_J)_{J \subseteq S}$ is a family of $\mathbb{Z}$-modules with $N_J \subseteq M_J$ for each J and $\phi_{JK} N_K \subseteq N_J$ for all J and K, then $\bigoplus_{J \subseteq S} \mathcal{A} N_J$ is an $\mathscr{H}$-submodule of the $\mathscr{H}$-module $\mathcal{A}\Gamma$.

In the context of our Magma programs, the transformations ϕ_{JK} above correspond to matrices. The function edgeMatrices computes them.

The required command is

```
ems:=edgeMatrices(cg4);
```

the output of edgeMatrices is a record with fields named gensets, sizes and mats. Here gensets is simply a list of all the descent sets that occur in the W-graph in question, and sizes is a list of the same length as gensets, giving the numbers of vertices with the various descent sets. (Thus the sum of the terms of the sequence sizes is the dimension of the $\mathscr{H}$-module.) The field mats is also a list of the same length as gensets. For each i, mats[i] is a list of pairs consisting of an integer j and a sizes[i]×sizes[j] matrix, which is the matrix corresponding to the transformation ϕ_{JK}, where J is gensets[i] and K is gensets[j]. All this really is just another way of presenting the W-graph.

Choose a subset J of S that occurs as a descent set of some vertex of Γ, and choose an element v of the $\mathbb{Z}$-module M_J. Initially, let $N_J = \mathbb{Z}v \subseteq M_J$ and let $N_L = \{0\}$ for all $L \neq J$. Whenever $L \not\subseteq J$, apply the transformation ϕ_{LJ} to v, and redefine N_L to be the $\mathbb{Z}$-module generated by N_L and $\phi_{LJ}v$. In other words, if $\phi_{LJ}v \in N_L$ then N_L is left unchanged, otherwise $\phi_{LJ}v$ is appended to the list of generators of N_L, producing a larger module. The process is repeated with v replaced by each newly found generator of each N_L, until no more new generators are found.

The function submod does this.

In the example above there are 40 descent sets. The command

```
print ems`sizes;
```

returns the sequence

```
[6,4,2,5,1,5,5,4,3,1,3,5,4,4,4,3,6,3,1,2,2,
          2,1,2,2,2,2,1,2,3,3,4,1,2,1,3,1,3,1,1]
```

and the command

```
print ems`gensets[40];
```

returns $\{1,3\}$. We see that there is just one vertex with descent set $\{1,3\}$. Now, for example, we could take $v = \gamma \in \Gamma_{\{1,3\}}$, and apply the function submod to produce a submodule of $\mathcal{A}\Gamma$. The required command is this:

```
xx:= submod([1],40,ems);
```

Here the third argument to submod is the W-graph, presented, as described above, as a record with fields gensets, sizes and mats. The output of submod will be a W-graph for a submodule, presented in the same way. The second argument to submod is a positive integer k less than or equal to the length of the sizes field of the third argument, and the first argument is an integer sequence of length sizes[k]. This sequence should be regarded as a vector in M_J, where J is gensets[k].

In our example, the variable xx corresponds to a W-graph for the irreducible 90-dimensional representation of E_6. The command

```
nwg:= emsToWG(xx);
```

converts it to a field with records basering, J, I and edges as previously described.

We also define functions for producing and checking matrix representations of Hecke algebras and Coxeter groups, given a W-graph. For example, the command

```
repH:= heckeAlgRep(nwg);
```

creates six 90×90 matrices, one for each of the generators of the Hecke algebra of type E_6. They can be printed via commands such as

```
print repH[1];
```

and so on. One can also tell Magma to check that the six matrices in question satisfy the defining relations of the Hecke algebra via the command

```
test("e6",repH);
```

Magma will print "true" after checking each relation.

One can also produce a representation of the Coxeter group rather than the Hecke algebra: the command

```
repW:= groupRep(nwg);
```

will do this. We have also defined a sequence of Magma commands which could be used to confirm that repW really is an irreducible representation of the Weyl group of type E_6.

4. The irreducible W-graphs for E_6

4.1. Minimal coset representatives and tables

The function inducеWGraph mentioned in the previous section implements the W-graph induction algorithm described in Section 1. As a first step it computes the minimal coset representatives for W_J in W and the action of the simple reflections on these. For example, there are 27 $(= \#W(E_6)/\#W(D_5) = \frac{51840}{1920})$ cosets of $W(D_5)$ in $W(E_6)$, and the 27 minimal coset representatives can be identified with sequences representing reduced words in the generators. (Of all possible reduced words for an element of W, we always use the one that is first in the lexicographic order.) In fact these sequences are as follows

```
[],
[ 6 ],
[ 6, 3 ],
[ 6, 3, 2 ],
[ 6, 3, 2, 1 ],
[ 6, 3, 2, 4 ],
[ 6, 3, 2, 1, 4 ],
[ 6, 3, 2, 4, 5 ],
[ 6, 3, 2, 1, 4, 2 ],
[ 6, 3, 2, 1, 4, 5 ],
[ 6, 3, 2, 1, 4, 2, 3 ],
[ 6, 3, 2, 1, 4, 2, 5 ],
[ 6, 3, 2, 1, 4, 2, 3, 5 ],
[ 6, 3, 2, 1, 4, 2, 3, 6 ],
[ 6, 3, 2, 1, 4, 2, 5, 4 ],
[ 6, 3, 2, 1, 4, 2, 3, 5, 4 ],
[ 6, 3, 2, 1, 4, 2, 3, 5, 6 ],
```

```
[ 6, 3, 2, 1, 4, 2, 3, 5, 4, 2 ],
[ 6, 3, 2, 1, 4, 2, 3, 5, 4, 6 ],
[ 6, 3, 2, 1, 4, 2, 3, 5, 4, 2, 1 ],
[ 6, 3, 2, 1, 4, 2, 3, 5, 4, 2, 6 ],
[ 6, 3, 2, 1, 4, 2, 3, 5, 4, 2, 1, 6 ],
[ 6, 3, 2, 1, 4, 2, 3, 5, 4, 2, 6, 3 ],
[ 6, 3, 2, 1, 4, 2, 3, 5, 4, 2, 1, 6, 3 ],
[ 6, 3, 2, 1, 4, 2, 3, 5, 4, 2, 1, 6, 3, 2 ],
[ 6, 3, 2, 1, 4, 2, 3, 5, 4, 2, 1, 6, 3, 2, 4 ],
[ 6, 3, 2, 1, 4, 2, 3, 5, 4, 2, 1, 6, 3, 2, 4, 5 ]
```

(where $W(D_5)$ is identified with the subgroup of $W(E_6)$ generated by the first five simple reflections).

For programming convenience we used right cosets rather than left cosets, despite the fact that the description presented in Section 1 uses left cosets. The program computes a table that describes what happens when the minimal coset representatives are multiplied by simple reflections. For D_5 in E_6 the table is as follows.

```
[-1,-1,-1,5,4,7,6,10,-4,8,-4,-4,-4,-4,-4,
        -4,-4,20,-4,18,22,21,24,23,-3,-3,-3],
[-2,-2,4,3,-1,-4,9,-4,7,12,-2,10,-2,-2,-5,
        18,-2,16,21,-4,19,-4,-3,25,24,-2,-2],
[-3,3,2,-2,-2,-2,-2,-2,11,-2,9,13,12,-3,16,
        15,-3,-5,-3,-5,23,24,21,22,-4,-4,-4],
[-4,-4,-4,6,7,4,5,-5,-1,-5,-1,15,16,-1,12,
        13,19,-2,17,-2,-2,-2,-2,-2,26,25,-1],
[-5,-5,-5,-5,-5,8,10,6,12,7,13,9,11,17,-1,
        -1,14,-1,-1,-1,-1,-1,-1,-1,-1,27,26],
[2,1,-3,-3,-3,-3,-3,-3,-3,-3,14,-3,17,11,-3,
        19,13,21,16,22,18,20,-5,-5,-5,-5,-5].
```

For example, the fifth sequence in this list describes the effect of appending a 5 to each of the 27 minimal coset representative sequences listed above, and then finding the coset representative corresponding to coset containing this word. Naming the coset representatives d_1 to d_{27}, the ith term of the fifth sequence says what happens when d_i is multiplied by $s = s_5$. For example, the 6th term of the sequence is 8, telling us that $d_6 s = d_8$. Naturally, this also means that $d_8 s = d_6$, and so the 8th term of the sequence should be 6, which it is. The values of i for which $d_i s$ is not a minimal coset representative correspond to the terms of the sequence that are negative. In these cases we must have $d_i s = s_j d_i$ for some j, and the corresponding term of the sequence is $-j$. Thus, for example, we have $d_i s_5 = s_1 d_i$ when i is 15, 16, 18, 19, 20, 21, 22, 23, 24 or 25.

With this information stored, it is in principle straightforward to apply the inductive formulas given in Section 1 to compute the generalized Kazhdan–Lusztig polynomials and hence the induced W-graph.

The character tables of groups $W(E_6)$ and $W(E_7)$ are taken from the paper of J.S. Frame [2]. We have also produced the tables such as: induced sign characters of type D_5 and the inner products $(\varphi, \chi|_{W(D_5)})$ where $\varphi \in \mathrm{Irr}(D_5)$ and $\chi \in \mathrm{Irr}(E_6)$. The information in the tables can be used to help verify that the W-graphs for E_6 produced by our Magma calculations are indeed correct.

Note that we have followed Frame's convention for the character tables of E_6 and E_7, writing the characters as columns rather than rows. Irreducible characters are given names such as d_a, d_b, etc., where d is the degree, and the subscripts distinguish different characters of the same degree. For example, the three irreducible characters of $W(E_6)$ of degree 60 are called 60_p, 60_n and 60_s. The subscripts p, q are used for characters that take positive values on the class of reflections, the subscripts n, m are used for those that take negative values on the class of reflections, while the subscript s is used for self-dual characters (vanishing on the class of reflections). Since the system GAP and Maple can also produce these tables, we do not reproduce them in this paper.

4.2. Cells in induced W-graphs

As remarked above, in most cases a W-graph corresponding to a given irreducible character of E_6 can be found as a cell in some W-graph induced from D_5. The table below describes exactly what happens in each case. For each irreducible character φ_i of $W(D_5)$ let $\mathrm{Ind}(\varphi_i)$ be the W-graph for E_6 induced from the W-graph corresponding to φ_i. Bracketed sums below correspond to cells. Thus, for example, inducing φ_{10} produces a graph with 3 cells, containing 60, 150 and 60 vertices (respectively). The two cells of size 60 correspond to the irreducible characters 60_p and 60_n, while the cell of size 150 corresponds to the reducible character with constituents 80_s, 60_s and 10_s.

$$
\begin{aligned}
\mathrm{Ind}(\varphi_1) &= 20_p + 6_p + 1_p;\\
\mathrm{Ind}(\varphi_2) &= 1_n + 6_n + 20_n;\\
\mathrm{Ind}(\varphi_3) &= 20_p + 64_p + 24_p;\\
\mathrm{Ind}(\varphi_4) &= 20_n + 64_n + 24_n;\\
\mathrm{Ind}(\varphi_5) &= 60_s + 60_p + 15_q;\\
\mathrm{Ind}(\varphi_6) &= 15_m + 60_s + 60_n;\\
\mathrm{Ind}(\varphi_7) &= 81_p + 81_n;\\
\mathrm{Ind}(\varphi_8) &= 6_p + (30_p + 15_p) + 20_p + 64_p;\\
\mathrm{Ind}(\varphi_9) &= 6_n + (15_n + 30_n) + 20_n + 64_n;\\
\mathrm{Ind}(\varphi_{10}) &= 60_p + (80_s + 60_s + 10_s) + 60_n;\\
\mathrm{Ind}(\varphi_{11}) &= 30_p + 64_p + 81_p + 60_p + (90_s + 80_s);\\
\mathrm{Ind}(\varphi_{12}) &= (90_s + 80_s) + 30_n + 64_n + 81_n + 60_n;\\
\mathrm{Ind}(\varphi_{13}) &= 20_p + (30_p + 15_q) + 64_p + 81_p + 60_p;\\
\mathrm{Ind}(\varphi_{14}) &= 15_p + 64_p + 81_p + (20_s + 90_s);\\
\mathrm{Ind}(\varphi_{15}) &= (20_s + 90_s) + 15_n + 64_n + 81_n;\\
\mathrm{Ind}(\varphi_{16}) &= 20_n + (30_n + 15_m) + 64_n + 81_n + 60_n;\\
\mathrm{Ind}(\varphi_{17}) &= 81_p + (90_s + 80_s + 60_s) + 64_n + 81_n + 24_n + 60_n;\\
\mathrm{Ind}(\varphi_{18}) &= 64_p + 81_p + 24_p + 60_p + (90_s + 80_s + 60_s) + 81_q.
\end{aligned}
$$

So the following statements hold:

1_p is a cell in $\mathrm{Ind}(\varphi_1)$;	1_n is a cell in $\mathrm{Ind}(\varphi_2)$;
6_p is a cell in $\mathrm{Ind}(\varphi_1)$;	6_n is a cell in $\mathrm{Ind}(\varphi_2)$;
15_p is a cell in $\mathrm{Ind}(\varphi_{14})$;	15_n is a cell in $\mathrm{Ind}(\varphi_{15})$;
15_q is a cell in $\mathrm{Ind}(\varphi_5)$;	15_m is a cell in $\mathrm{Ind}(\varphi_6)$;
20_p is a cell in $\mathrm{Ind}(\varphi_3)$;	20_n is a cell in $\mathrm{Ind}(\varphi_4)$;
30_p is a cell in $\mathrm{Ind}(\varphi_{11})$;	30_n is a cell in $\mathrm{Ind}(\varphi_{12})$;
64_p is a cell in $\mathrm{Ind}(\varphi_{14})$;	64_n is a cell in $\mathrm{Ind}(\varphi_{15})$;
81_p is a cell in $\mathrm{Ind}(\varphi_7)$;	81_n is a cell in $\mathrm{Ind}(\varphi_7)$;
24_p is a cell in $\mathrm{Ind}(\varphi_3)$;	24_n is a cell in $\mathrm{Ind}(\varphi_4)$;
60_p is a cell in $\mathrm{Ind}(\varphi_3)$;	60_n is a cell in $\mathrm{Ind}(\varphi_3)$;
60_s is a cell in $\mathrm{Ind}(\varphi_6)$.	

Thus we may construct W-graphs for all the irreducible characters except 90_s, 80_s, 20_s and 10_s. Of course, some of the others can be found in several different ways. For example, the W-graph corresponding to character 20_p can be obtained as a cell in any one of the following: $\mathrm{Ind}(\varphi_1)$, $\mathrm{Ind}(\varphi_8)$, $\mathrm{Ind}(\varphi_{13})$.

4.3. Special cases

For the remaining four self dual characters we can obtain W-graphs by using the method described in Section 3 above.

10_s. The 10-dimensional representation (corresponding to the character 10_s) of E_6 can be constructed easily enough by hand. It is an extension of one of the 10-dimensional irreducibles of D_5 (corresponding to the character φ_{10}). The E_6 W-graph is the same as the D_5 W-graph with the 6th generator of E_6 added to some of the descent sets. It can also be extracted from a cell of degree 150 in the representation of E_6 induced from φ_{10}. In this cell there are 3 vertices with descent set $\{2\}$. Calling the corresponding basis vectors x, y, z (labelled appropriately) the vector $x+y-z$ generates the 10-dimensional submodule (this was found by trial and error). The following sequence of Magma commands could be used.

90_s. The 90-dimensional representation was constructed as follows. The W-graph corresponding to the character φ_{15} of $W(D_5)$ was induced to E_6 and the cells were found. There is a cell of degree 110, having a single vertex with descent set $\{3,5\}$. The submodule generated by the corresponding basis vector gave the 90-dimensional submodule.

80_s. The 80-dimensional representation was constructed as follows. The W-graph corresponding to the character φ_{10} of D_5 was induced to E_6 and the cells were found. There is a cell of degree 150 having a single vertex with descent set $\{3,5,6\}$. The submodule generated by the corresponding vector gave the 80-dimensional irreducible.

20_s. The 20-dimensional representation corresponding to character 20_s can be constructed easily by hand, since it corresponds to the exterior cube of the reflection representation.

In the Coxeter diagram of E_6, we denote the six basis vectors of the reflection module V as $v_1,\dots,v_6$. The exterior cube $\bigwedge^3 V$ has dimension 20; it has a basis consisting of all elements $v_{ijk} = v_i \wedge v_j \wedge v_k$, where $1 \leqslant i < j < k \leqslant 6$. The v_{ijk} can be identified with the vertices of a W-graph, the descent set of v_{ijk} being $\{i,j,k\}$. The action of the generators on each such vector gives the corresponding edges. Now, for example, since

$$s_5 v_1 = v_1,$$

$$s_5 v_3 = v_3,$$

$$s_5 v_4 = v_4 + v_5$$

we find that

$$s_5(v_{134}) = v_1 \wedge v_3 \wedge (v_4 + v_5) = v_1 \wedge v_3 \wedge v_4 + v_1 \wedge v_3 \wedge v_5 = v_{134} + v_{135}.$$

We conclude that there is an edge joining v_{134} and v_{135} with $\mu(v_{135}, v_{134}) = 1$, the other edges can be found similarly.

Using Magma programs, the same W-graph can also be extracted from the cell of degree 110 in $\mathrm{Ind}(\varphi_{15})$. There are 3 vertices with descent set $\{2,4,5\}$, and if the corresponding basis vectors are denoted by x, y and z (in an appropriate order) then $x-y+z$ generates the 20-dimensional submodule.

5. The construction of irreducible W-graphs for type E_7

The group $W(E_7)$ is the direct product of its derived group and a group or order 2. The character table of the derived group was calculated by Frame [2]. These characters can be identified with the characters of $W(E_7)$ whose kernels contain the longest element w_S. Following Frame's notation, each of these characters is given a name of the form d_a, d_b or d_c, where d is the degree. (For example, the two of the degrees 21_a and 21_b.) Extending Frame's notation in a natural way, we give the dual characters (whose kernels do not contain w_S) names of the form d_x, d_y or d_z (with d_x dual to d_a, etc.).

Similarly to the case of E_6 above, we have the following decompositions of the W-graphs induced from the irreducibles of E_6:

$$
\begin{aligned}
\mathrm{Ind}(1_p) &= 21_y + 27_a + 7_x + 1_a;\\
\mathrm{Ind}(1_n) &= 1_x + 7_a + 27_x + 21_y;\\
\mathrm{Ind}(6_p) &= (105_x + 120_a) + (21_a + 56_x) + 27_a + 7_x;\\
\mathrm{Ind}(6_n) &= (105_a + 120_x) + (21_x + 56_a) + 27_x + 7_a;\\
\mathrm{Ind}(10_s) &= 70_a + 210_b + 210_y + 70_x;\\
\mathrm{Ind}(15_p) &= 189_a + (280_x + 35_x) + 210_a + 105_x + 21_a;\\
\mathrm{Ind}(15_n) &= 189_x + (280_a + 35_a) + 210_x + 105_a + 21_x;\\
\mathrm{Ind}(15_q) &= 216_x + 280_b + 105_b + 189_y + 15_x + 35_b;\\
\mathrm{Ind}(15_m) &= 216_a + 280_y + 105_y + 189_b + 15_a + 35_y;\\
\mathrm{Ind}(20_p) &= 189_z + 168_a + 210_a + 189_y + (120_a + 105_x) + 21_y + (35_b + 56_x) + 27_a;\\
\mathrm{Ind}(20_n) &= 189_c + 168_x + 210_x + 189_b + (120_x + 105_a) + 21_b + (35_y + 56_a) + 27_x;\\
\mathrm{Ind}(20_s) &= 189_x + 35_a + 336_a + 336_x + 189_a + 35_x;\\
\mathrm{Ind}(24_p) &= 105_z + (420_a + 84_a) + 378_x + 189_z + 168_a;\\
\mathrm{Ind}(24_n) &= 105_c + (420_x + 84_x) + 378_a + 189_c + 168_x;\\
\mathrm{Ind}(30_p) &= 189_y + 105_b + 405_a + 120_a + 210_x + (315_x + 280_x) + 56_x;\\
\mathrm{Ind}(30_n) &= 189_b + 105_y + 405_x + 120_x + 210_x + (315_a + 280_a) + 56_a;\\
\mathrm{Ind}(60_p) &= (70_x + 280_b + 315_x) + (216_x + 405_a) + 189_y + 105_b + 168_a + (512_x + 512_a) + 210_b;\\
\mathrm{Ind}(60_n) &= (70_a + 280_y + 315_a) + (216_a + 405_x) + 189_b + 105_y + 168_x + (512_x + 512_a) + 210_y;\\
\mathrm{Ind}(60_s) &= 280_b + 378_x + 216_a + 210_b + (512_a + 512_x) + 84_x + 210_y + 84_a + 378_a + 216_x + 280_y;\\
\mathrm{Ind}(64_p) &= (336_x + 420_a) + 378_x + (189_a + 405_a) + 189_y + 168_a\\
&\quad + (315_x + 280_x + 280_b) + 210_a + 189_z + (105_x + 120_a);\\
\mathrm{Ind}(64_n) &= (336_a + 420)_x + 378_a + (189_x + 405_x) + 189_b + 168_x\\
&\quad + (315_a + 280_a + 280_y) + 210_x + 189_c + (105_a + 120_x);\\
\mathrm{Ind}(80_s) &= 315_a + 405_a + 378_x + 420_a + 210_b + (512_a + 512_x) + 210_y\\
&\quad + 420_x + 378_a + 405_x + 315_x;\\
\mathrm{Ind}(81_p) &= (336_x + 420_a) + (512_a + 512_x) + (336_a + 420_x) + 189_z\\
&\quad + (216_x + 405_a) + 105_c + (280_b + 280_x + 315_x) + 210_a;\\
\mathrm{Ind}(81_n) &= (336_a + 420_x) + (512_a + 512_x) + (336_x + 420_a) + 189_c\\
&\quad + (216_a + 405_x) + 105_z + (280_y + 280_a + 315_a) + 210_x;\\
\mathrm{Ind}(90_s) &= 280_a + 189_a + 405_a + 378_a + (336_x + 420_a) + 378_x\\
&\quad + (336_a + 420_x) + (512_a + 512_x) + 189_x + 405_x + 280_x.
\end{aligned}
$$

With the exception of the two characters of degree 512, every irreducible occurs as a cell in some induced representation. Specifically, the following statements hold.

1_a is a cell of $\mathrm{Ind}(1_p)$;	1_x is a cell of $\mathrm{Ind}(1_n)$;
7_a is a cell of $\mathrm{Ind}(6_n)$;	7_x is a cell of $\mathrm{Ind}(6_p)$;
27_a is a cell of $\mathrm{Ind}(1_p)$;	27_x is a cell of $\mathrm{Ind}(1_n)$;
21_a is a cell of $\mathrm{Ind}(15_p)$;	21_x is a cell of $\mathrm{Ind}(15_n)$;
21_b is a cell of $\mathrm{Ind}(20_n)$;	21_y is a cell of $\mathrm{Ind}(20_p)$;
35_a is a cell of $\mathrm{Ind}(20_s)$;	35_x is a cell of $\mathrm{Ind}(20_s)$;
35_b is a cell of $\mathrm{Ind}(15_q)$;	35_y is a cell of $\mathrm{Ind}(15_m)$;
105_a is a cell of $\mathrm{Ind}(15_n)$;	105_x is a cell of $\mathrm{Ind}(15_p)$;
105_b is a cell of $\mathrm{Ind}(15_q)$;	105_y is a cell of $\mathrm{Ind}(15_m)$;
105_c is a cell of $\mathrm{Ind}(24_n)$;	105_z is a cell of $\mathrm{Ind}(24_p)$;
189_a is a cell of $\mathrm{Ind}(15_p)$;	189_x is a cell of $\mathrm{Ind}(15_n)$;
189_b is a cell of $\mathrm{Ind}(30_n)$;	189_y is a cell of $\mathrm{Ind}(30_p)$;
189_c is a cell of $\mathrm{Ind}(24_n)$;	189_z is a cell of $\mathrm{Ind}(24_p)$;
15_a is a cell of $\mathrm{Ind}(15_m)$;	15_x is a cell of $\mathrm{Ind}(15_q)$;
315_a is a cell of $\mathrm{Ind}(80_s)$;	315_x is a cell of $\mathrm{Ind}(80_s)$;
405_a is a cell of $\mathrm{Ind}(30_p)$;	405_x is a cell of $\mathrm{Ind}(30_n)$;
168_a is a cell of $\mathrm{Ind}(20_p)$;	168_x is a cell of $\mathrm{Ind}(20_n)$;
56_a is a cell of $\mathrm{Ind}(30_n)$;	56_x is a cell of $\mathrm{Ind}(30_p)$;

120_a is a cell of $\mathrm{Ind}(30_p)$;	120_x is a cell of $\mathrm{Ind}(30_n)$;
210_a is a cell of $\mathrm{Ind}(15_p)$;	210_x is a cell of $\mathrm{Ind}(15_n)$;
210_b is a cell of $\mathrm{Ind}(60_p)$;	210_y is a cell of $\mathrm{Ind}(60_n)$;
280_a is a cell of $\mathrm{Ind}(90_s)$;	280_x is a cell of $\mathrm{Ind}(90_s)$;
280_b is a cell of $\mathrm{Ind}(15_q)$;	280_y is a cell of $\mathrm{Ind}(15_m)$;
336_a is a cell of $\mathrm{Ind}(20_s)$;	336_x is a cell of $\mathrm{Ind}(20_s)$;
216_a is a cell of $\mathrm{Ind}(15_m)$;	216_x is a cell of $\mathrm{Ind}(15_q)$;
378_a is a cell of $\mathrm{Ind}(64_n)$;	378_x is a cell of $\mathrm{Ind}(64_p)$;
84_a is a cell of $\mathrm{Ind}(60_s)$;	84_x is a cell of $\mathrm{Ind}(60_s)$;
420_a is a cell of $\mathrm{Ind}(80_s)$;	420_x is a cell of $\mathrm{Ind}(80_s)$;
70_a is a cell of $\mathrm{Ind}(10_s)$;	70_x is a cell of $\mathrm{Ind}(10_s)$.

The 512-dimensional representations were constructed as follows.

The W-graph corresponding to the character 60_s of E_6 was induced to E_7. There is a cell of degree 1024, having two vertices with descent set $\{2\}$. Denote the basis vectors by x and y. The submodules generated by the vectors $x+y$ and $x-y$ gave the two 512-dimensional submodules.

5.1. Postscript

The methods of the present paper have now been used to produce W-graphs for the for the irreducible representations of E_8; these have been incorporated into "Chevie" (www.math.rwth-aachen.de/~CHEVIE/chevie-gap.html).

We also understand that Hiroshi Naruse has calculated W-graphs for the irreducible representations of F_4 and E_6.

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